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OPTIMAL CONTROL OF SUPERSONIC INLETS TO MINIMIZE UNSTARTS

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SUMMARY

This report describes a preliminary investigation into the use of optimal control theory for the design of controls for a supersonic inlet. In particular, the control of a mixed-compression supersonic inlet is formulated as a linear stochastic optimal control and estimation problem. An inlet can exhibit an undesirable instability (unstart) due to excessive inlet normal shock motion. For the optimal control formulation of the inlet program, a nonquadratic performance index is used, which is equal to the expected frequency of inlet unstarts. The computer program developed in this report determines optimal controllers which minimize this physically meaningful performance index.

A linear lumped parameter, single-input, single-output inlet model is considered. The disturbance considered is a white Gaussian airflow perturbation at the compressor face station. The measured variable is normal shock position, corrupted by additive Gaussian white measurement noise. A Kalman filter is used to generate the estimates of the state variables. State estimate feedback gains are found by solving a series of quadratic control problems (Riccati equation) using the quadratic equivalence principle. The variation of the performance index with shock position tolerance, plant disturbance level, and measurement noise level is investigated. Results are presented for different levels of available control effort.

INTRODUCTION

For optimum performance of present day supersonic aircraft, it is necessary to provide an efficient means for decelerating the air from supersonic to subsonic velocity before it enters the compressor of the turbojet engine. A common method for accomplishing this is a variable geometry supersonic inlet as shown in figure 1. The inlet shown is an axisymmetric mixed compression type to be used on future supersonic aircraft. In normal operation, air enters the inlet past a weak oblique shock wave and is compressed supersonically past the minimum area station (throat) up to the position of

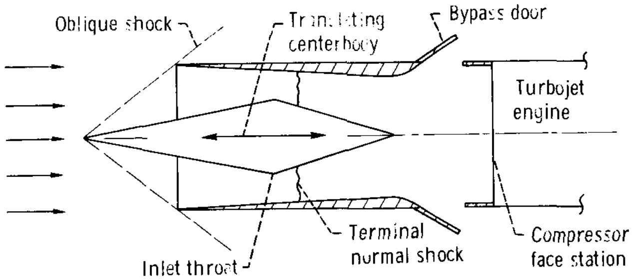

Figure 1. - Schematic of supersonic axisymmetric inlet.

the terminal normal shock. After the normal shock, the air is further compressed subsonically until it reaches the inlet of the turbojet engine (compressor face). For efficient operation, the inlet should maximize the recovered pressure at the compressor face over a wide range of operating conditions and maintain a stable flow pattern in the presence of external disturbances.

The flow configuration of this type of inlet is bistable; the desired started condition just described and a second unstarted condition having a strong external shock system, subsonic flow in the initial convergent section, poor pressure recovery, and sometimes oscillatory flow (buzz). The desired started condition becomes unstable and switches to unstart when either of two events occur:

(1) A decrease in upstream Mach number causes the throat Mach number to decrease to one. In this case, a second (unstable) normal shock forms at the throat and moves rapidly upstream, out of the inlet.

(2) A decrease in compressor face airflow causes the terminal normal shock to move upstream of the throat. Here again, the shock moves rapidly upstream, out of the inlet.

The severe reduction in compressor face airflow and pressure occurring after an unstart may cause a compressor stall or combustor flame-out. In addition, the increased nacelle drag on an unstarted inlet may cause the aircraft to yaw suddenly. Also, a started inlet with normal shock too far downstream of the throat will have relatively poor pressure recovery and undesirable flow distortion at the engine face. Control is, therefore, required to maintain throat Mach number and shock position within acceptable limits to provide stable, efficient inlet operation.

The primary inlet control variables are overboard bypass doors and a translating or collapsing inlet centerbody. Opening the bypass doors allows air to be dumped overboard, causing the shock to move downstream away from the throat. The translating centerbody varies the throat area, thereby varying the throat Mach number. A combination of these two control modes is used to ensure stable inlet operation in the face of both upstream and downstream disturbances.

The inlet can encounter both random and deterministic disturbances in a typical flight environment. Deterministic disturbances are those such as atmospheric pressure changes due to a shock wave from a passing aircraft, pilot induced engine transients, aircraft maneuvers, and engine compressor stall. Disturbances which are random in nature are those such as atmospheric gusts, atmospheric temperature changes, combustion noise (in a duct burning fan) fed back from the engine, and aerodynamic compressor noise.

Present inlet control systems (refs. 1 and 2) have been designed to minimize system response to deterministic disturbances such as ramp or sinusoidal changes in upstream Mach number and/or engine air flow rate. These systems were evaluated by examining the shape of their frequency response curves. Another approach which has been taken is to minimize system response to random disturbances. This was used by Barry (ref. 3) for the case where the random disturbance was atmospheric turbulence. He developed a method for predicting the expected number of unstarts per flight mile. Inlet controls were then evaluated on the basis of their effectiveness in minimizing unstart frequency.

The approach described in this report is an extension of Barry's work to the problem of designing optimal controllers which minimize unstart frequency. Basically, the inlet control problem is formulated as a linear stochastic optimal control problem using Barry's expected frequency of unstart equation as the performance index. This optimal design method is direct in that it leads to a control system which minimizes a physically meaningful performance index rather than some intermediate quantity such as the area enclosed by the closed loop frequency response plot or shock position deviation. This performance index also leads to controllers uniquely different from controllers which, for instance, minimize shock position deviation.

OPTIMAL CONTROL FORMULATION

Basic Formulation

Bypass door control is used to avoid unstarts due to inlet disturbances which occur at the compressor face. Fast-acting bypass doors, such as described in references 1 and 4, have been used to regulate against this type of disturbance. Also, to avoid unstarts due to atmospheric disturbances, the inlet throat area must be controlled. Typically, this is done by translating the centerbody, which can only be done relatively slowly. However, the throat Mach number responds very rapidly to upstream (atmospheric) disturbances, due to the fact that flow is supersonic between the entrance to the inlet and the throat. Thus, it is difficult to mechanize a throat area control that will effectively reduce unstarts caused by upstream disturbances. For this reason, initial

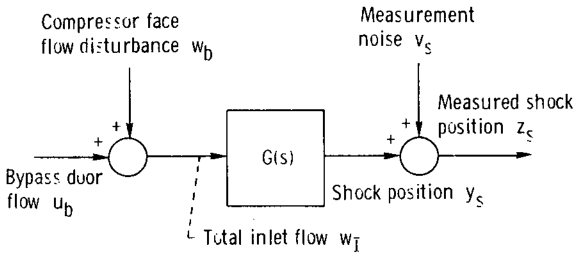

Figure 2. - Single-input, single-output inlet control.

studies have concentrated on the optimal control of bypass doors to reduce unstarts due to downstream disturbances only.

Figure 2 is a block diagram of the inlet chosen for this study. Conventional control design for this inlet terminated with a long pipe ending in a choke plate was discussed in reference 1. For this report, an inlet transfer function model was obtained for the inlet terminated with a choke plate located at the compressor face station. It was found that, by curve fitting experimental frequency response data (unpublished NASA data), a good fit could be obtained over the range from 0 to 100 hertz using the form (Symbols are defined in the appendix.)

$$G(s) = \frac{y_s(s)}{w_I(s)} = \frac{K_I\left(\frac{s}{a} + 1\right)e^{-\tau_I s}}{\left(\frac{s}{b} + 1\right)\left[\left(\frac{s}{\omega_n}\right)^2 + \frac{2\zeta s}{\omega_n} + 1\right]} \tag{1}$$

The dead time in $G(s)$ is characteristic of inlet dynamics. However, in order to make use of the well-developed theory of linear stochastic optimal control and estimation (e.g., ref. 5), a finite state (lumped parameter) model was needed. Thus, at the outset, the dead time was approximated by a third-order Padé, resulting in a sixth-order state variable model for the inlet.

In figure 2, the disturbance to the inlet w_b represents a random air flow perturbation at the compressor face. It is assumed to be white Gaussian noise with a constant power spectral density (psd) equal to q. The output variable y_s (shock position) is measured through a noisy channel with measurement noise v_s. The noise is assumed to be white Gaussian with a psd of r.

The following performance index was chosen to be minimized:

$$J = \lambda + k\sigma_{u_b}^2 \tag{2}$$

where

$$\lambda = \frac{1}{2\pi}\sqrt{\frac{\sigma_{\dot{y}_s}^2}{\sigma_{y_s}^2}}\exp\left(-\frac{\alpha^2}{2\sigma_{y_s}^2}\right) \tag{3}$$

and

λ expected frequency of inlet unstarts

k positive weighting factor

$\sigma_{u_b}^2$ mean square value of bypass door flow rate

$\sigma_{\dot{y}_s}^2$ mean square value of shock velocity

$\sigma_{y_s}^2$ mean square value of shock position

α shock position tolerance (distance between undisturbed shock position and inlet throat)

J was selected so that the control must minimize unstarts λ while limiting the amount of bypass door flow σ_{u_b} needed to do so.

The λ relation (eq. (3)) is a classical exceedance equation given in reference 3. It gives the expected frequency with which the Gaussian random variable y_s exceeds the level α in the positive direction. Weighting factor k for $\sigma_{u_b}^2$ is selected to penalize the control variable so that the level of control effort won't exceed that available. Selection of the control effort weighting factor k will be discussed in a later section.

In order to write equation (3) for the expected frequency of unstarts, the following assumptions must be made: (1) the inlet disturbances are Gaussian, (2) the inlet dynamics are linear, and (3) the controller is restricted to be linear and time-invariant.

If a general approach to the optimal control problem were chosen (via dynamic programming, for example), the controller thereby obtained would, in general, be nonlinear due to the nonquadratic nature of the performance index of equation (2). This would be a violation of assumption (3) mentioned in the previous paragraph and, hence, the computation of λ by equation (3) would be invalid. One way around this problem is to restrict the controller to be linear and find the optimum linear, time-invariant controller which minimizes the expected frequency of unstarts. This is essentially what was done in this study.

It is known that the solution to the linear stochastic optimal control problem for quadratic performance indices dictates linear feedback of the optimal estimates of the state variables. This fact is a result of the so-called separation theorem (ref. 5). The solution technique can be extended to systems with nonquadratic performance indices by employing the quadratic equivalence principle developed by Edinger, Skelton, Stone, Ward, and White (ref. 6). The result is a linear control law which minimizes a quadratic approximation to the nonquadratic performance index. The method of quadratic equivalence was chosen for use in the inlet control problem because it forces a linear solution. In addition, the computer programs required are those that have been fairly well developed for solving the linear optimal control problem for quadratic performance indices.

Linear Stochastic Optimal Control and Estimation Problem Solution

Before dealing with the nonquadratic problem, it is well to review the linear stochastic optimal control and estimation problem. It can be outlined as follows (ref. 5). Given a plant described by

$$x = A(t)x + B(t)u + w \tag{4}$$

where x is the state vector, u is the control vector, and w is the plant disturbance vector. An output vector y is defined as

$$y = H(t)x \tag{5}$$

and a measurement vector z is given as

$$z = H(t)x + v \tag{6}$$

where v is the measurement noise vector. Both w and v are white zero mean Gaussian, and uncorrelated with each other, having correlation matrices defined by

$$E\left\{w(t)w^T(t + \tau)\right\} = Q(t)\delta(\tau) \tag{7a}$$

$$E\left\{v(t)v^T(t + \tau)\right\} = R(t)\delta(\tau) \tag{7b}$$

A(t), B(t), and H(t) are real, possibly time varying, matrices. The initial state vector x(0) is assumed to be random with a covariance

$$E\left\{x(0)x^T(0)\right\} = P_0 \tag{8}$$

The problem is to minimize the quadratic performance index

$$C = E\left\{\frac{1}{2}\, x^T(t_f)S_{t_f}\, x(t_f) + \frac{1}{2}\int_0^{t_f}\left[x^T Q_c(t)x + 2x^T N(t)u + u^T P_c(t)u\right]dt\right\} \tag{9}$$

Matrices S_f, $Q_c(t)$, and $N(t)$ are at least positive semidefinite, and $P_c(t)$ is a positive definite matrix.

The known solution to this problem (using the separation theorem) is

$$u = -K_c(t)\hat{x} \tag{10}$$

where $\hat{x}$ is the optimal estimate of the state vector x and is generated with a Kalman filter described by

$$\dot{\hat{x}} = A(t)\hat{x} + B(t)u + K_e(t)\left[z - H(t)\hat{x}\right] \tag{11}$$

The optimal control gains $K_c(t)$ are given by

$$K_c(t) = P_c^{-1}(t)\left[B^T(t)S(t) + N^T(t)\right] \tag{12}$$

where $S(t)$ is the positive definite symmetric solution to the following matrix Riccati equation:

$$\left.\begin{aligned} -\dot{S} &= S\left(A - BP_c^{-1}N^T\right) + \left(A - BP_c^{-1}N^T\right)^T S - S\left(BP_c^{-1}B^T\right)S + \left(Q_c - NP_c^{-1}N^T\right) \\ S(t_f) &= S_{t_f} \end{aligned}\right\} \tag{13}$$

(For simplicity, the explicit time dependence of matrices A, B, etc. has been dropped.) The Kalman filter gains $K_e(t)$ are given by

$$K_e(t) = P(t)H(t)^T R(t)^{-1} \tag{14}$$

where $P(t)$ is a positive definite symmetric solution to a Riccati equation, which is

$$\left.\begin{aligned} \dot{P} &= AP + PA^T - P(H^T R^{-1} H)P + Q \\ P(0) &= P_0 \end{aligned}\right\} \tag{15}$$

$P(t)$ is also the covariance of the error in the estimate $\hat{x}$.

Equations (10) to (15) then define the solution to the problem of controlling a linear plant to minimize a quadratic performance index when the plant is subject to white noise disturbances. Figure 3 is a block diagram of the solution to the optimal control and estimation problem showing the state estimator and state estimate feedback. The state estimator (Kalman filter, eq. (11)) is basically a model of the plant driven by control u and measurement z. Signal z is compared with $\hat{z}$, the estimated measurement, to form a term which is the error in the estimate of the measurement. This error is then multiplied by Kalman filter gains K_e and added back into the filter as a "correction" term. The filter output $\hat{x}$ is weighted by the control gains $K_c(t)$ to form the optimal

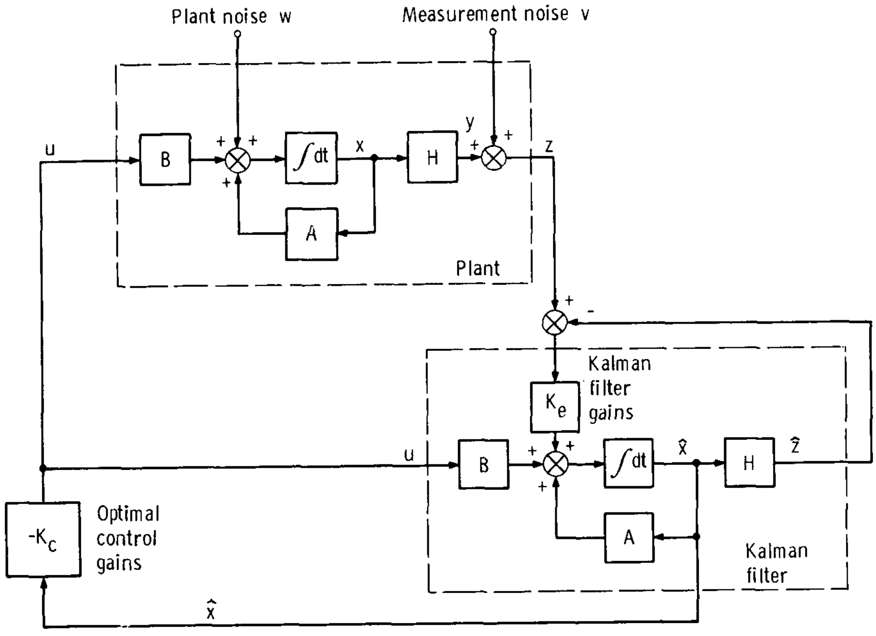

Figure 3. - Combined optimal regulator-state estimator.

control vector u. In general, time varying control gain matrix $K_c(t)$ is calculated off-line by integrating equation (13) backward in time from $t = t_f$ to zero with the initial condition $S(t_f) = S_{t_f}$. Similarly, time varying filter gains $K_e(t)$ are calculated off-line by integrating equation (15) forward in time, with the initial condition P_0 being the a priori estimate of the covariance of x.

Constant Gain Solution

In selecting the nonquadratic performance index (eq. (2)), one assumes constant mean square values for all variables, this implying a constant gain system. That is, controller gains for the nonquadratic problem must be time invariant. For the quadratic problem, assuming matrices A, B, H, Q_c, N, and P_c are constant, control gains K_c are constant for times sufficiently less than t_f (which can be considered very large). K_c is then given by

$$K_c = P_c^{-1}(B^T S + N^T) \tag{16}$$

where S is the steady-state solution of the Riccati equation (eq. (13)). If, in addition, Q and R are constant matrices, Kalman filter gains K_e become constant for time sufficiently greater than zero (so that initial condition effects have died out). K_e is then given by

$$K_e = PH^T R^{-1} \tag{17}$$

where P is the steady-state solution of the Riccati equation (eq. (15)). The optimal system will then have all gains (K_e and K_c) constant for times $0 \ll t \ll t_f$. During this time, all expected values are constant.

Rewriting equation (9) as follows:

$$C = E\left\{\frac{1}{2} x^T(t_f) S_f x(t_f)\right\} + \frac{1}{2}\int_0^{t_f} E\left\{x^T Q_c x + 2x^T N u + u^T P_c u\right\} dt \tag{18a}$$

it is seen that, for $t_f \to \infty$, the major contribution to the index is the constant value of the integrand, obtained for $t \gg 0$ (after all initial condition effects have died out). Thus, the constant gain control derived can be said to have minimized that constant value given by

$$C_1 = E\left\{x^T Q_c x + 2x^T Nu + u^T P_c u\right\} \tag{18b}$$

If the performance index for the inlet problem were purely quadratic, the solution to the control and estimation problem would be given by equations (16) and (17). However, such is not the case. For the inlet control problem, a nonquadratic performance index defined by equation (2) is to be minimized. The following section will show how the principle of quadratic equivalence can be used to yield a solution which is a linear feedback of state estimates identical to that of figure 3.

Implementing Quadratic Equivalence

For the quadratic problem, the necessary condition for the existence of a stationary point (such as a minimum) is that the variation of the performance index equal zero. First, define the scalars $u \triangleq u_b$, $w \triangleq w_b$, and $y \triangleq y_s$. Then, taking the variation of equation (18b) and setting it equal to zero, the following equation can be obtained:

$$\delta E\left\{x^T Q_c x\right\} + 2\delta E\left\{x^T N u_b\right\} + \delta E\left\{u_b^T P_c u_b\right\} = 0 \tag{19}$$

The variation of the nonquadratic performance index can be put in the form of equation (19) by use of the quadratic equivalence principle. Then the known solution for the infinite time, linear, quadratic problem just discussed can be adopted. This can be done as follows.

First, the variation of J (eq. (2)) is set equal to zero; that is,

$$\delta\left[\frac{1}{2\pi}\sqrt{\frac{\sigma_{\dot{y}_s}^2}{\sigma_{y_s}^2}}\exp\left(\frac{-\alpha^2}{2\sigma_{y_s}^2}\right) + k\sigma_{u_b}^2\right] = 0 \tag{20}$$

Then equation (20) is expanded in terms of $\sigma_{y_s}^2$, $\sigma_{\dot{y}_s}^2$, and $\sigma_{u_b}^2$ giving

$$\delta\left(\sigma_{\dot{y}_s}^2\right) + W_1\delta\left(\sigma_{y_s}^2\right) + W_2\delta\left(\sigma_{u_b}^2\right) = 0 \tag{21}$$

where W_1 and W_2 are defined as

$$W_1 = \frac{\sigma^2_{\dot{y}_s}}{\sigma^2_{y_s}}\left(\frac{\alpha^2}{\sigma^2_{y_s}} - 1\right) \tag{22a}$$

$$W_2 = 4\pi k\sigma_{y_s}\sigma_{\dot{y}_s}\exp\left(\frac{\alpha^2}{2\sigma^2_{y_s}}\right) \tag{22b}$$

In order to compare equation (21) with equation (19), it is necessary to first express $\sigma^2_{\dot{y}_s}$ and $\sigma^2_{y_s}$ in terms of x, y, u, A, B, and H by using equations (4) and (5). The mean square value of the output y_s written as an expected value, is

$$\sigma^2_{y_s} = E\left\{y_s^2\right\} \tag{23}$$

but

$$y_s = Hx \tag{24}$$

thus,

$$\sigma^2_{y_s} = E\left\{x^T H^T Hx\right\} \tag{25}$$

Since

$$\dot{y}_s = H\dot{x} \tag{26}$$

then

$$E\left\{\dot{y}_s^2\right\} = E\left\{(H\dot{x})^T(H\dot{x})\right\} \tag{27}$$

Thus, the mean square value of $\dot{y}_s$ becomes

$$\sigma^2_{\dot{y}_s} = E\left\{\dot{y}_s^2\right\} = E\left\{(HAx + HBu_b + Hw_b)^T(HAx + HBu_b + Hw_b)\right\} \tag{28}$$

The nature of the plant must be such that the product Hw_b is equal to zero; otherwise, $\sigma_{\dot{y}_s}$ would be infinite and, hence, it would be impossible to minimize J. (This requirement is satisfied if the plant transfer function has more poles than zeros.) With $Hw_b = 0$,

$$\sigma^2_{\dot{y}_s} = E\left\{x^T A H^T H A x + 2x^T A^T H^T H B u_b + u_b^T B^T H^T H B u_b\right\} \tag{29}$$

since u_b is a scalar.

Also, $\sigma^2_{u_b}$, as an expected value, can be written

$$\sigma^2_{u_b} = E\left\{u_b u_b^T\right\} = E\left\{u_b^T u_b\right\} \tag{30}$$

Now, $\sigma^2_{y_s}$, $\sigma^2_{\dot{y}_s}$, and $\sigma^2_{u_b}$ from equations (25), (29), and (30) can be substituted into equation (21) to obtain

$$\delta E\left\{x^T A^T H A x + 2x^T A^T H^T H B u_b + u_b^T B^T H^T H B u_b\right\} + W_1 \delta E\left\{x^T H^T H x\right\} + W_2 \delta E\left\{u_b u_b^T\right\} = 0 \tag{31}$$

Collecting terms results in equation (31) becoming

$$\delta E\left\{x^T\left(A^T H^T H A + W_1 H^T H\right)x\right\} + \delta E\left\{2x^T\left(A^T H^T H B\right)u_b\right\} + \delta E\left\{u_b^T\left(B^T H^T H B + W_2\right)u_b\right\} = 0 \tag{32}$$

Compare term by term equation (32), obtained from the nonquadratic performance index J, with equation (19), obtained from the quadratic performance C_1. The following relations can then be made:

$$\left.\begin{aligned} Q_c &= A^T H^T H A + W_1 H^T H \\ N &= A^T H^T H B \\ P_c &= B^T H^T H B + W_2 \end{aligned}\right\} \tag{33}$$

The nonquadratic problem can now be solved by solving the quadratic problem (eqs. (13) and (15)) with Q_c, N, and P_c defined by equation (33), and $\dot{P}$ and $\dot{S}$ set equal to zero. Equivalence, however, only is assured at the optimum. At this point W_1 and W_2 must satisfy equations (22a) and (22b) such that quadratic equivalence is maintained. Since W_1 and W_2 are defined in terms of mean square values such as $\sigma^2_{y_s}$ and $\sigma^2_{\dot{y}_s}$, the technique by which these values are determined is discussed in the next section.

Mean Square Values

The mean square values $\sigma^2_{y_s}$ and $\sigma^2_{\dot{y}_s}$ are functions of P, the covariance of the error in the estimate of the states, and X, the covariance of the states. Since the error in the estimate is

$$e \triangleq \hat{x} - x \tag{34}$$

the covariance of x can be written as

$$X = E\left\{xx^T\right\} = E\left\{(\hat{x} - e)(\hat{x} - e)^T\right\} = \hat{X} + P \tag{35}$$

since $E\left\{\hat{x}e^T\right\} = 0$. Similarly,

$$\dot{X} = \dot{\hat{X}} + \dot{P} \tag{36}$$

where $\dot{\hat{X}}$ is derived in Bryson (ref. 5, p. 417) as

$$\dot{\hat{X}} = (A - BK_c)\hat{X} + \hat{X}(A - BK_c)^T + K_eRK_e^T \tag{37}$$

Using this value for $\dot{\hat{X}}$ and obtaining $\dot{P}$ from equation (15), X becomes

$$\dot{X} = (BK_c)P + P(BK_c)^T + Q + (A - BK_c)X + X(A - BK_c)^T \tag{38}$$

For the inlet problem, only the steady-state value of X is needed; thus, $\dot{X} = 0$ and equation (38) becomes

$$(BK_c)P + P(BK_c)^T + Q + (A - BK_c)X + X(A - BK_c)^T = 0 \tag{39}$$

the steady-state covariance matrix equation.

Now, $\sigma^2_{y_s}$ and $\sigma^2_{\dot{y}_s}$ must be expressed in terms of P and X. From equation (25),

$$\sigma^2_{y_s} = E\left\{x^T H^T H x\right\} = E\left\{H x x^T H^T\right\} = HE\left\{x x^T\right\}H^T = HXH^T \tag{40}$$

In a similar manner, equation (28) can be rearranged in terms of $E\left\{xx^T\right\}$, $E\left\{xu_b^T\right\}$, and $E\left\{u_b u_b^T\right\}$ giving

$$\sigma^2_{\dot{y}} = HAE\left\{xx^T\right\}A^T H^T + 2HAE\left\{xu_b^T\right\}B^T H^T + HBE\left\{u_b u_b^T\right\}B^T H^T \tag{41}$$

But it can be shown that

$$E\left\{xu_b^T\right\} = -(X - P)K_c^T \tag{42}$$

and

$$E\left\{u_b u_b^T\right\} = \sigma^2_{u_b} = K_c(X - P)K_c^T = K_c\hat{X}K_c^T \tag{43}$$

Substituting these results into equation (41), $\sigma^2_{\dot{y}}$ becomes

$$\sigma^2_{\dot{y}_s} = HAXA^T H^T - 2HA(X - P)K_c^T B^T H^T + HBK_c(X - P)K_c^T B^T H^T \tag{44}$$

Thus W_1 and W_2 can be calculated as functions of X and P using equations (40) and (44). In addition, mean square control effort $\sigma^2_{u_b}$ is a simple function of $\hat{X}$, given by equation (43).

This completes the derivation of the equations necessary to design an optimal inlet control system to minimize unstarts. Next, the solution of these equations will be described.

CONTROL COMPUTATION PROCEDURE

First, the estimation problem was solved, yielding a set of estimator gains K_e. Then the quadratic control problem was solved for a number of trial (W_1, W_2) pairs.

The solution to each quadratic problem K_c was used with the K_e previously determined to compute the mean square values of the state vector. From this, $\sigma^2_{y_s}$, $\sigma^2_{\dot{y}_s}$, and $\sigma^2_{u_b}$ were calculated. For a range of k's, J, λ, W_1, and W_2 were then calculated. All these data were scanned off-line to determine optimum points. This was done by first finding points of minimum J for each value of k. As a check, an alternate method was used. This was to find points for constant k where the assumed W_1 was sufficiently close to the calculated W_1 and the assumed W_2 was close to the calculated W_2. The accuracy with which the W_1 and W_2 equations (eqs. (22a) and (22b)) were satisfied

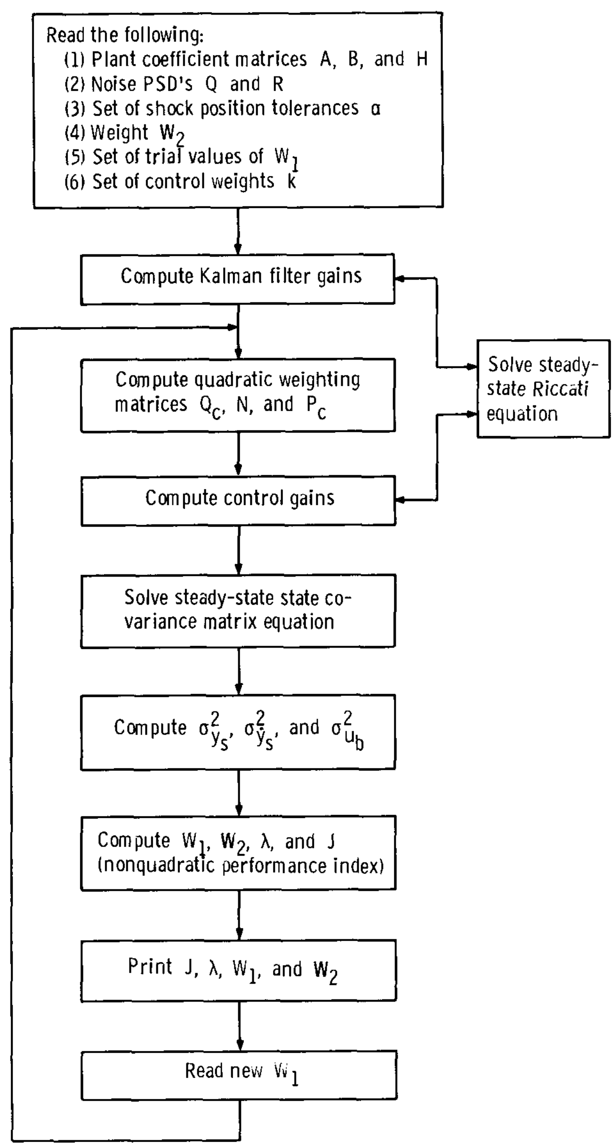

Figure 4. - Control computation procedure flow chart.

was a function of the number of trial pairs (W_1,W_2). That is, finding the optimum more accurately required more pairs.

The digital computer program used in solving the inlet control design problem is outlined in figure 4. Inputs to the program are plant coefficient matrices A, B, and H, noise spectral densities Q and R, trial quadratic weights W_1 and W_2, mean square control weight k, and output variable (shock position) tolerance α.

The first step in the solution procedure is to compute the Kalman filter gains, which requires the steady-state solution of a Riccati equation. (Note that the Kalman filter gains need only be calculated once for a given pair of noise psd's.) Next, the control weighting matrices Q_c, N, and P_c are found. Using these weights, the control problem is then solved for the optimum feedback gains, where again a Riccati equation is solved. The mean square system behavior is then determined by solving the steady-state state covariance matrix equation. Then the mean square values of y, $\dot{y}$, and u can be found. The final calculations, using $\sigma^2_{y_s}$, $\sigma^2_{\dot{y}_s}$, and $\sigma^2_{u_b}$, are W_1, W_2, λ, and J, which are all printed out.

The two major subroutines used are one for the steady-state Riccati equation and one for the steady-state state covariance matrix equation. The steady-state Riccati equation subroutine used the negative exponential method (ref. 7). For the inlet problem, this method proved much faster than a direct fifth-order Runge-Kutta integration. The state covariance matrix equation was first solved by integration. However, since only the steady-state solution was needed, it was found that a faster method was to transform equation (39) to a set of $n(n + 1)/2$ linear equations. Equation (39) is also a form of the so-called Lyapunov matrix equation. The solution method chosen was only one of a number of possible alternate methods for solving the Lyapunov equation (ref. 8).

RESULTS AND DISCUSSION

Introduction

The preceding synthesis procedure was applied to develop an optimal estimator and control gains for the inlet described in equation (1). To do this, a sixth-order inlet model was obtained by combining a third-order dead time approximation with the third-order transfer function of equation (1).

Parameters obtained from the curve fit using G(s) given by equation (1) are as follows:

$$K_I = 5.59 \text{ cm/(kg/sec)}$$

$$a = 210 \text{ rad/sec}$$

$$b = 80 \text{ rad/sec}$$

$$\omega_n = 365 \text{ rad/sec}$$

$$\zeta = 0.30$$

$$\tau_I = 0.004 \text{ sec}$$

For a dead time of 0.004 second, the third-order Padé is

$$e^{-0.004\,s} = \frac{-5.333\times10^{-10}\, s^3 + 1.60\times10^{-6}\, s^2 - 2.0\times10^{-3}\, s + 1}{5.333\times10^{-10}\, s^3 + 1.60\times10^{-6}\, s^2 + 2.0\times10^{-3}\, s + 1} \tag{45}$$

Substituting for the dead time in equation (1) from equation (45) and using the aforementioned constants, the sixth-order inlet transfer function becomes

$$G(s) = \frac{y_s(s)}{w_I(s)} = \frac{(5.59)(2.001\times10^{16})(-2.537\times10^{-12}\, s^4 + 7.083\times10^{-9}\, s^3 - 7.920\times10^{-6}\, s^2 + 2.760\times10^{-3}\, s + 1)}{s^6 + 3.30\times10^{3}\, s^5 + 4.80\times10^{6}\, s^4 + 3.460\times10^{9}\, s^3 + 1.160\times10^{12}\, s^2 + 3.230\times10^{14}\, s + 2.001\times10^{16}} \tag{46}$$

Using equation (46), the A, B, and H matrices for the state variable representation were conveniently obtained in phase variable form. But to improve the accuracy of the Riccati equation solutions, the problem was time scaled by 200 to 1, resulting in the following A, B, and H matrices:

$$A = \begin{bmatrix} 0.0 & 1.0 & 0.0 & --- & --- & 0.0 \\ 0.0 & --- & 1.0 & --- & & --- \\ --- & & & 1.0 & & \\ & & & & 1.0 & --- \\ --- & & & & --- & 1.0 \\ -312.7 & -1009 & -725.0 & -432.5 & -120.0 & -16.50 \end{bmatrix}$$

$$B^T = [0,\ 0,\ 0,\ 0,\ 0,\ 1.0]$$

$$H = \left\{5.59\,[\mathrm{cm}/(\mathrm{kg}/\mathrm{sec})]\right\} \cdot [312.7,\ 172.6,\ -99.06,\ 17.71,\ -1.269,\ 0.0]$$

The plant disturbance vector was defined as

$$w^T = [0,\ 0,\ 0,\ 0,\ 0,\ w_b]$$

and its correlation matrix was

$$E\left\{w(t)w^T(t+\tau)\right\} = Q\delta(\tau)$$

where

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & - & - & & & - \\ 0 & - & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & - & - & - & - & q \end{bmatrix}$$

and q is the psd of w_b. Solutions to the estimation and control problem were obtained for a plant noise psd of

$$q = 1.03\times10^{-3}\ (\mathrm{kg}/\mathrm{sec})^2/(\mathrm{rad}/\mathrm{sec})$$

and two typical values of measurement noise v_s psd's

$$r = 3.22\times10^{-5} \quad \text{and} \quad 3.22\times10^{-6}\ \mathrm{cm}^2/(\mathrm{rad}/\mathrm{sec})$$

Note that, since w_b and v_s both have flat spectra (their psd's not being functions of frequency), q and r do not change with time scaling. However, to compute λ (unstart frequency) in real time units, $\sigma^2_{y_s}$ and $\sigma^2_{\dot{y}_s}$ had to be expressed in real time. Actually, only $\sigma^2_{\dot{y}_s}$ needed rescaling. This is because

$$\sigma^2_{\dot{y}_s}(t) = E\left\{\left[\frac{dy_s(t)}{dt}\right]^2\right\} \tag{47}$$

and for $\tau = \gamma t$, $d/dt = \gamma(d/d\tau)$, and $\sigma^2_{\dot{y}_s}(t)$ becomes

$$\sigma^2_{\dot{y}_s}(t) = E\left\{\left[\gamma \frac{dy_s}{d\tau}(\tau)\right]^2\right\}$$

$$= \gamma^2 E\left\{\left[\frac{dy_s}{d\tau}(\tau)\right]^2\right\}$$

$$= \gamma^2 \sigma^2_{\dot{y}_s}(\tau) \tag{48}$$

Thus, for $\gamma = 200$, $\sigma^2_{\dot{y}_s}(\tau)$ from the program was multiplied by $(200)^2$ before λ was calculated.

A Kalman filter was designed for each q, r pair. Then a family of optimum controllers was designed by varying control weight k. The members of the family differ in that although all are optimum, each requires a different rms value of the control variable σ_{u_b}. Knowing σ_{u_b} is important because it dictates how large the control actuator capacity must be. Conversely, if the maximum available control variable level is known, the controller can be designed to avoid control saturation. Saturation is undesirable first because, if it occurs, the system becomes nonlinear and the whole optimum controller design procedure is in question. In addition, reliability is decreased if control saturation occurs. For example, in the case of the inlet, high deceleration forces are imposed on the bypass doors whenever they are driven full open or closed, which could lead to premature actuator failure. With these facts available, the results of the design program can be used to select a controller that will both be optimal and avoid saturation.

Shock Position Tolerance Sensitivity

The calculated optimal inlet control problem solutions can best be displayed by relating unstarts, control effort, and shock position tolerance for a family of optimal control systems. Figure 5 shows how well one family of optimum controllers is able to minimize unstarts. The expected frequency of unstarts λ is plotted as a function of

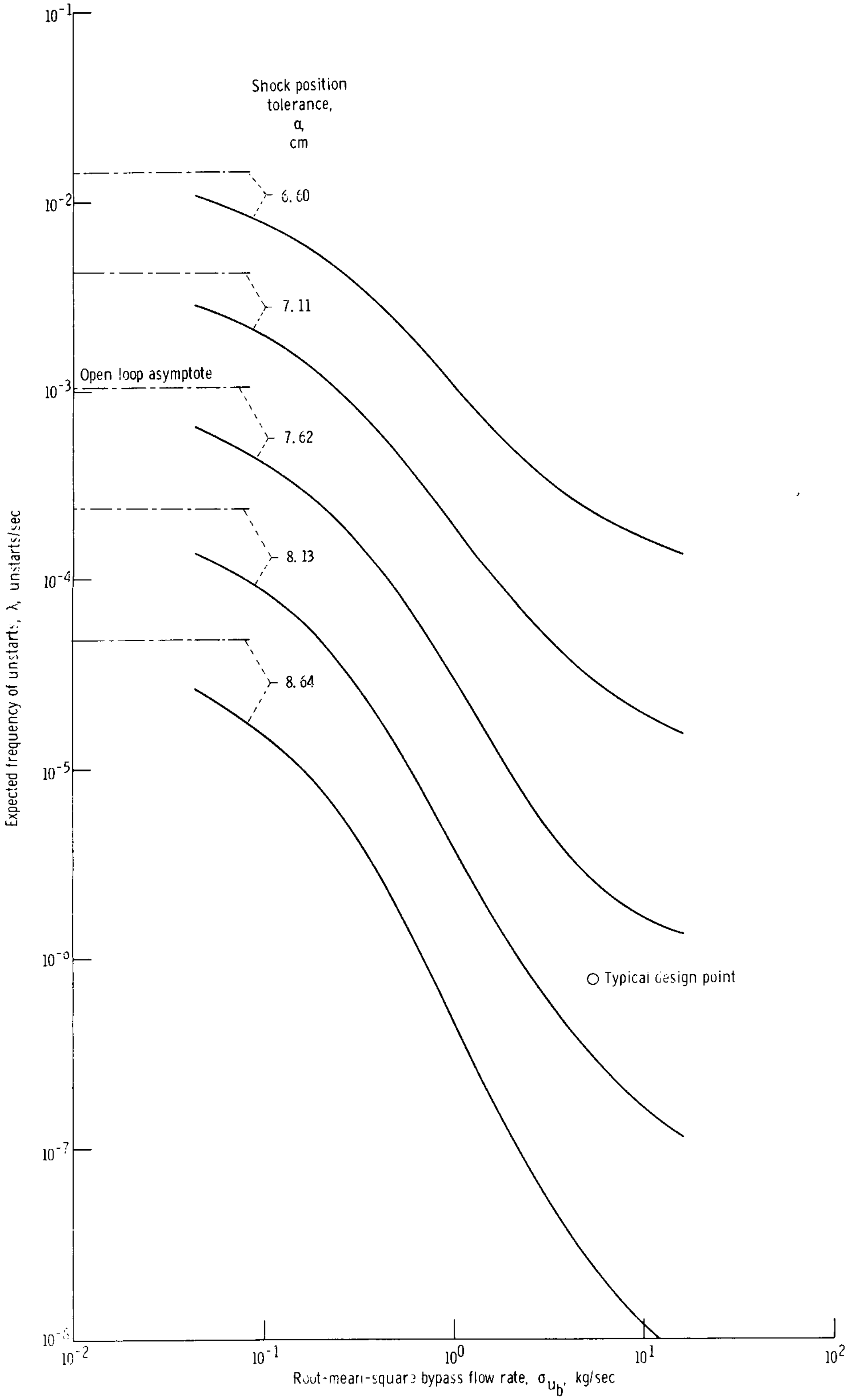

Figure 5. - Expected frequency of unstarts as a function of bypass flow for various shock position tolerances. Power spectral density of compressor face disturbance flow rate $q = 1.03\times10^{-3}$ (kg/sec)^2/rad/sec; power spectral density of shock position measurement noise $r = 3.21\times10^{-5}$ cm^2/rad/sec; total inlet flow rate $w_I = 16.2$ kilograms per second.

rms bypass flow rate σ_{u_b} for a typical range of shock position tolerance α. Each value of σ_{u_b} corresponds to a different controller (K_c gains), using the same Kalman filter (K_e gains) to generate the state estimates. Open loop values of λ for different α's are shown along the vertical axis so that the improvement gained by optimal control can be seen. Note that, as σ_{u_b} goes to zero, the curves asymptotically approach their open loop values, as would be expected. Also, as α increases, λ decreases for both closed and open loop cases.

A more conventional way of contrasting open and closed loop system performance is by comparing rms values. However, for the inlet problem, all disturbances have been assumed white and thus have infinite rms values. But if plant and measurement noise are assumed to be not white but colored, their mean square values become finite. Then various signals can be compared to them on an rms basis. For the following discussion, assume that the spectral densities of w_b (compressor face disturbance) and v_s (shock position measurement noise) are flat to 2000 radians per second and fall off on a log-log plot at a two-to-one slope thereafter (first-order coloring). With such a high rolloff frequency, the noise will still appear essentially white to the inlet. The mean square value of w_b can be calculated by integrating the psd of w_b. For first-order coloring, mean square value of w_b turns out to be simply (ref. 9)

$$\sigma_{w_b}^2 = \frac{q\omega_c}{2} \tag{49}$$

where q is the zero-frequency value of the psd and ω_c is the corner (rolloff) frequency of the coloring.

The inlet being used has a nominal captured flow rate w_I of 16.2 kilograms per second at Mach 2.5. For a q of 1.03×10^{-3} $(kg/sec)^2/(rad/sec)$ and ω_c of 2000 radians per second, the mean square value of w_b is 1.02 kilograms per second or 6.3 percent of nominal inlet captured flow. The open loop rms shock excursion for this disturbance level is then 1.65 centimeters. For a low frequency measurement noise psd level of 3.21×10^{-5} $cm^2/(rad/sec)$ (shown in fig. 5), the rms noise-to-signal ratio for open loop shock position is about 11 percent. With control, rms shock position ranges from 1.65 centimeters down to 1.14 centimeters for the maximum value of σ_{u_b} shown. Although this change is rather small, the corresponding change in λ is large - about three orders of magnitude. This is because σ_{y_s} (rms shock position) appears in the exponential term in λ, thus λ is quite sensitive to it.

Note that as shock position tolerance increases, for a constant value of σ_{u_b}, the difference between closed and open loop λ's increases. This indicates control is more effective for large values of shock position tolerance. But larger λ's cause lower inlet pressure recovery; thus there is a definite limit as to how large α can be. A trade-off then exists between λ and α (pressure recovery).

The other main control design consideration is what value of σ_{u_b} is acceptable; that is, how large can σ_{u_b} be without having saturation occur. There is no completely satisfactory answer to this question. The only sure solution is to simulate the system designed for a specific σ_{u_b}, drive it with the desired random disturbances, and see whether or not saturation occurs. Newton, Gould, and Kaiser (ref. 10) suggest, as a rule of thumb, σ_{u_b} be no more than one-third of the saturation value to minimize the possibility of control variable saturation.

For the inlet studied, maximum bypass flow is about equal to nominal capture flow. Using Newton's rule, a suggested value of σ_{u_b} would be 5.4 kilograms per second. As a design example, assume that it is desired that mean distance between unstarts be 10^6 kilometers. For a cruise Mach number of 2.5, this means that λ must be 7.5×10^{-7} unstarts per second. Using figure 5, for σ_{u_b} chosen to be 5.4 kilograms per second, shock position tolerance should be about 8 centimeters (see circular data point on fig. 5). Alternately, the inlet designer may insist some maximum value of α not be exceeded so that sufficient pressure recovery is insured. In this case, figure 5 can be used to predict the expected distance between unstarts for some maximum σ_{u_b} and α.

Measurement Noise Level Sensitivity

In addition to shock position tolerance, other parameters which strongly influence the expected frequency of unstarts are plant noise psd q and measurement noise psd r. This follows from the fact that the ratio of q/r dictates the Kalman filter gains. The change in filter characteristics alters mean square values and thus λ changes. Plant noise psd q is usually specified by the environment and cannot usually be changed by the designer. However, r is partly a function of the noise of the sensor and its associated instrumentation amplifier so the designer may be able to influence this variable.

Figure 6 illustrates the effect a variation in r has on λ. Plotted are three curves (solid lines) from figure 5 for α = 6.60, 7.62, and 8.64 centimeters plus curves (dashed lines) for lower measurement noise (where r is decreased by a factor of ten). The rms level of the low measurement noise is 3.5 percent of open loop shock position rms compared to the 11 percent mentioned previously for the high measurement noise case. Measurement noise psd r has a strong effect on λ, especially at higher rms control levels. For example, if, as in the previous example, a 10^{-6}-kilometer-per-unstart

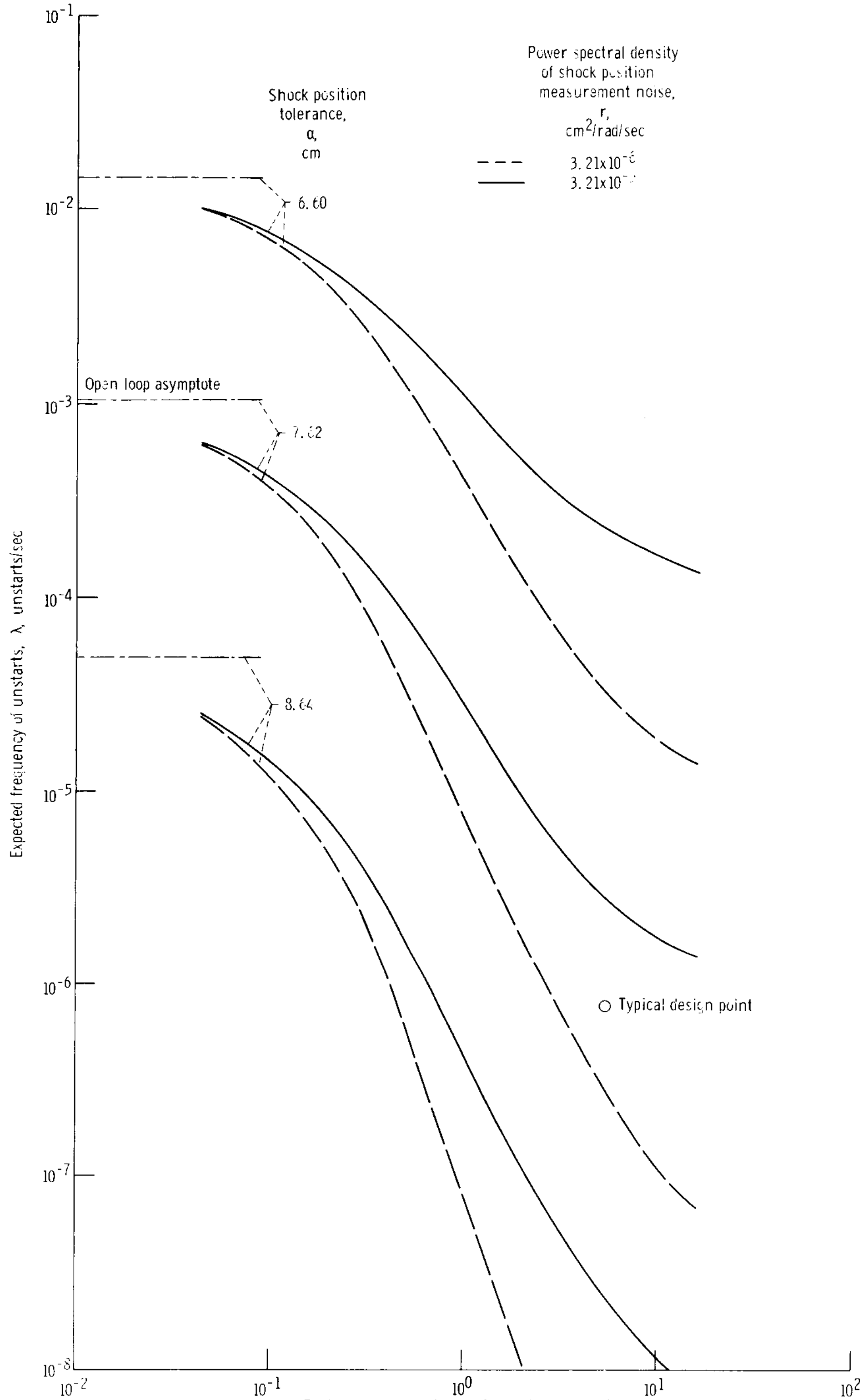

Figure 6. - Expected frequency of unstarts as a function of bypass flow for various shock position tolerances and measurement noise levels. Power spectral density of compressor face disturbance flow rate $q = 1.03\times10^{-3}$ $(kg/sec)^2$/rad/sec; total inlet flow rate w_I = 16.2 kilograms per second.

figure is desired, it can be achieved with α of about 7.4 centimeters instead of 8 centimeters if measurement noise is low. This means reliability can be held constant while pressure recovery is increased, if r can be reduced. Thus benefits can be gained by designing low noise transducers and amplifiers since they can lead to a smaller r.

CONCLUDING REMARKS

This report has described formulating the problem of controlling a supersonic inlet as an optimal control problem. A physically meaningful nonquadratic performance index, the expected frequency of unstarts, was used in the formulation. A single-input, single-output inlet control case was solved via quadratic equivalence by solving a series of quadratic estimation-control problems. The method provides a straightforward procedure for determining a set of linear feedback controls which minimize the nonquadratic performance index. Using this method for control design, the expected frequency of inlet unstart can be minimized. The effects of measurement noise and shock position tolerance were shown to have a large influence on the performance index.

The theory used in this report can be extended to multivariable systems. Thus, assuming a suitable analytical model is available, control of both bypass doors and throat area, with downstream and upstream disturbances, can be included. Provisions can be made to allow coloring of both plant and measurement noise. The performance index can be expanded to allow for more than one mode of unstart. Also, more practical considerations can be introduced into the problem, such as how to minimize sensitivity of the state estimates to inaccuracies in the inlet model and prevent consequent degradation in performance, inclusion of actuator dynamic limitations, use of other available measurements (such as compressor face pressure) to improve performance, techniques for simplifying the Kalman filter, and ways to handle dead time directly without introducing Padé approximations.

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APPENDIX - SYMBOLS

A	system matrix, $n \times n$
a	transfer function zero, rad/sec
B	control matrix, $n \times c$
b	transfer function pole, rad/sec
C	quadratic performance index
C_1	time-invariant quadratic performance index
c	integer, number of control inputs
$E\{\ \}$	expected value of
e	error in state vector estimate, $n \times 1$
$G(s)$	inlet transfer function, cm/(kg/sec)
H	measurement matrix, $m \times n$
J	nonquadratic performance index
K_c	control gain matrix, $c \times n$
K_e	estimator gain matrix, $n \times m$
K_I	inlet gain, cm/(kg/sec)
k	positive weighting factor
m	integer, number of measurements
N	quadratic weighting matrix on state control, $n \times c$
n	integer, number of state variables
P	solution of estimator matrix Riccati equation, $n \times n$
P_c	quadratic weighting matrix on control, $c \times c$
P_0	initial value of P, $n \times n$
Q	power spectral density matrix of plant disturbance, $n \times n$
Q_c	quadratic weighting matrix on state, $n \times n$
q	power spectral density of w_b, $(\text{kg/sec})^2/(\text{rad/sec})$
R	power spectral density of measurement noise, $m \times m$
r	power spectral density of v_s, $\text{cm}^2/(\text{rad/sec})$
S	solution of control matrix Riccati equation, $n \times n$

S_f	terminal quadratic weighting matrix, $n \times n$
s	Laplace variable, sec^{-1}
t	time
t_f	final time
u	control vector, $c \times 1$
u_b	control bypass door flow rate, kg/sec
v	measurement noise vector, $m \times 1$
v_s	shock position measurement noise, cm
W_1	equivalence coefficient
W_2	equivalence coefficient
w	plant disturbance vector, $n \times 1$
w_b	compressor face disturbance flow rate, kg/sec
w_I	total inlet flow rate, kg/sec
X	covariance matrix of x, $n \times n$
$\hat{X}$	covariance matrix of $\hat{x}$, $n \times n$
x	state vector, $n \times 1$
$\hat{x}$	estimate of state vector, $n \times 1$
y	output vector, $m \times 1$
y_s	shock position, cm
z	measurement vector, $m \times 1$
$\hat{z}$	estimated measurement vector, $m \times 1$
z_s	measured shock position, cm
α	shock position tolerance, cm
γ	time scaling factor
$\delta(\ )$	first variation of
$\delta(\tau)$	delta function
ζ	damping ratio
λ	expected frequency of unstarts, unstarts/sec
$\sigma_{u_b}^2$	mean square bypass flow rate, $(kg/sec)^2$

$\sigma^2_{w_b}$ mean square disturbance flow rate, $(kg/sec)^2$

$\sigma^2_{y_s}$ mean square shock position, cm^2

$\sigma^2_{\dot{y}_s}$ mean square shock velocity, $(cm/sec)^2$

τ time, sec

τ_I inlet dead time, sec

ω_c noise cutoff frequency, rad/sec

ω_n natural frequency, rad/sec

Superscripts:

T matrix transpose

differentiation with respect to time

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16. Abstract

This report describes a method for designing supersonic inlet controls based on a desire to minimize inlet unstart. The design problem is formulated as one in linear stochastic optimal control and estimation. However, the performance index chosen (to be minimized) is the expected frequency of unstarts. Since this index is nonquadratic, the principle of quadratic equivalence is applied, so that the control consists of linear state variable feedback. Estimates of unmeasurable and/or noisy states required for control are then generated using a Kalman filter. Results show the sensitivity of unstart frequency to nominal normal shock position, control bypass door capacity, and measurement noise level, for white noise disturbances.

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